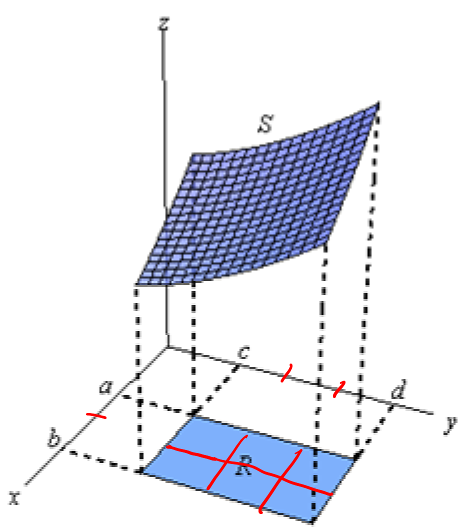
Section 15.1 Double Integrals over Rectangles

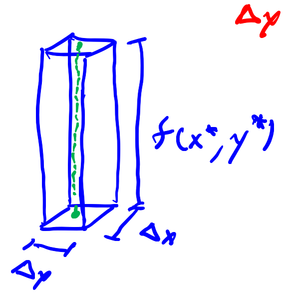
Let f be a continuous function over the rectangle $R = [a,b]\times[c,d]$. We first suppose that $f(x,y) \geq 0$. The graph of f is a surface with equation $z = f(x,y)$. Let S be the solid that lies above R and under the graph of f, that is,

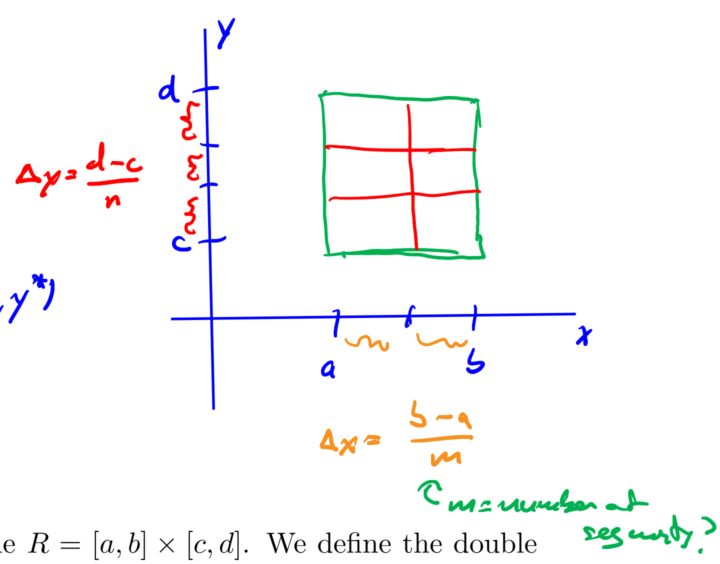
$$S = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq z \leq f(x,y),\ (x,y) \in R\}$$

- The first step is to divide the rectangle R into subrectangles. We accomplish this by dividing the interval $[a,b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b-a)/m$ and dividing the interval $[c,d]$ into n subintervals $[y_{i-1}, y_i]$ of equal width $\Delta y = (c-d)/n$.
- By drawing lines parallel to the coordinate axes through the endpoints of these intervals, we form subrectangles $R_{i,j}$ each of area $\Delta A = \Delta x \Delta y$.
- If we choose a sample point $(x^*_{i,j}, y^*_{i,j})$ in each $R_{i,j}$, then we can approximate the part of S that lies above each $R_{i,j}$ by a thin rectangular box with base $R_{i,j}$ and height $f(x^*_{i,j}, y^*_{i,j})$. The volume of this box is given by
$$f(x^*_{i,j}, y^*_{i,j})\Delta A$$
- If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation for the total volume of S:

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{i,j}, y^*_{i,j})\ \Delta A$$







DEF: Let f be a continuous function over the rectangle $R = [a,b]\times[c,d]$. We define the double integral of f over R as

$$\iint_R f(x,y)\ dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{i,j}, y^*_{i,j})\ \Delta A$$

if this limit exists.

Midpoint Rule for Double Integrals

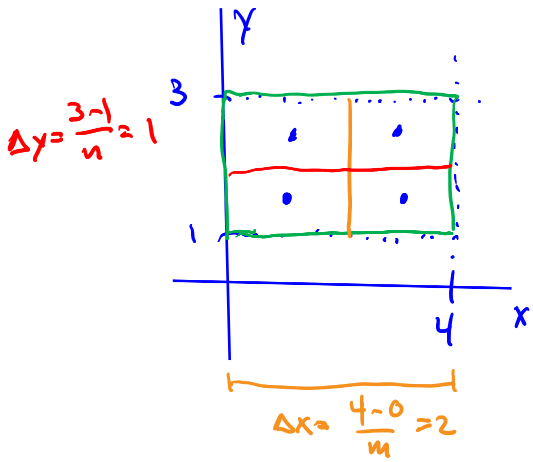
$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i,\overline{y}_j)\,\Delta A$$

where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{i-1}, y_i]$.

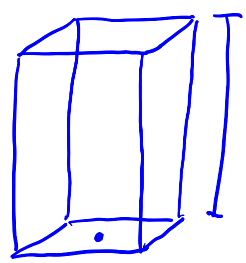
Ex1. Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral

$$\iint_R \underbrace{(2x^2 + y^2)}_{f(x,y)}\,dA$$

where $R = \overbrace{[0,4]}^{x} \times \overbrace{[1,3]}^{y}$. (Set up only)



$\Delta A = (\Delta x)(\Delta y) = (2)(1) = 2$



$$\iint_R (2x^2+y^2)\,dA \approx (\Delta A) f(1,1.5) + (\Delta A) f(3,1.5) + (\Delta A) f(1,2.5) + (\Delta A) f(3,2.5)$$

$$\approx (\Delta A)\left[f(1,1.5) + f(3,1.5) + f(1,2.5) + f(3,2.5)\right]$$

$$\approx 2\left[(2(1)^2 + (1.5)^2) + (2(3)^2 + 1.5^2) + (2(1) + 2.5^2) + (2(3)^2 + 2.5^2)\right]$$

("slices of bread")

Sections 15.1 - 15.2: Double Integrals over General Regions

DEF. Let $z = f(x,y)$ be a continuous function of two variables on the region $\mathcal{R}$.

$$\iint\limits_{\mathcal{R}} f(x,y)\ dA := \int_{x_{\min}}^{x_{\max}} \Big[\int_{y_{\text{bott}}(x)}^{y_{\text{top}}(x)} f(x,y)\ dy\Big] dx$$

"Iterated Integrals"

Ex2. Compute $\iint\limits_{\mathcal{R}} f(x,y)\ dA$ if $f(x,y) = \dfrac{y}{x^5+1}$, and $\mathcal{R} = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x^2\}$.

$$\iint_R f(x,y)\,dA = \int_0^1 \left(\int_0^{x^2} \frac{y}{x^5+1}\,dy\right)dx$$

Inner: $\displaystyle\int_0^{x^2} \frac{y}{x^5+1}\,dy = \underbrace{\frac{1}{x^5+1}}_{\text{constant}} \int_0^{x^2} y\,dy = \frac{1}{x^5+1}\left(\frac{y^2}{2}\Bigg|_{y=0}^{y=x^2}\right)$

$$= \frac{1}{x^5+1}\left(\frac{x^4}{2} - \frac{0}{2}\right) = \frac{x^4}{2(x^5+1)}$$

$$\int \frac{x^4}{2(x^5+1)}\,dx = \int \frac{\cancel{x^4}}{2(u)}\,\frac{du}{5\cancel{x^4}} = \int \frac{1}{10u}\,du = \frac{1}{10}\ln|u| + C$$

$$= \frac{1}{10}\ln|x^5+1| + C$$

$u = x^5+1$

$du = 5x^4\,dx$

Outer: $\displaystyle\int_0^1 \left(\frac{x^4}{2(x^5+1)}\right)dx = \left(\frac{1}{10}\ln|x^5+1|\right)\Bigg|_{x=0}^{x=1}$

$$= \frac{1}{10}\ln|2| - \cancel{\frac{1}{10}\ln|1|}^{\,0}$$

$$= \frac{\ln(2)}{10}$$

so $\displaystyle\iint_R f(x,y)\,dA = \frac{\ln(2)}{10}.$

Fubini's Theorem: If $f(x,y)$ is continuous on the region $\mathcal{R}$, then

$$\iint\limits_{\mathcal{R}} f(x,y)\;dA = \int_{x_{\min}}^{x_{\max}} \Big[\int_{y_{\text{bott}}(x)}^{y_{\text{top}}(x)} f(x,y)\;dy\Big]dx = \int_{y_{\min}}^{y_{\max}} \Big[\int_{x_{\text{left}}(y)}^{x_{\text{right}}(y)} f(x,y)\;dx\Big]dy$$

This means that we can exchange the order of integration and get the same answer.

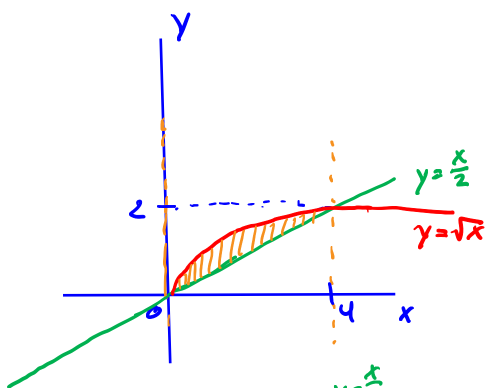
CAUTION: If we change the order of integration, we need to find "new" limits of integration.

Ex3. It is known that

$$I = \int_0^4 \int_{x/2}^{\sqrt{x}} \{4y+2\}\;dy\;dx = 8$$

Sketch the region of integration for the integral and write an equivalent integral with the order of integration reversed.

$$\begin{cases} 0 \le x \le 4 \\ \underbrace{\tfrac{x}{2}}_{y=\frac{x}{2}} \le y \le \underbrace{\sqrt{x}}_{y=\sqrt{x}} \end{cases}$$



$$I = \int_0^2 \Big(\int_{y^2}^{2y} 4y+2 \; dx \Big) \; dy$$

$y = \frac{x}{2}$

$x = y^2$
$y = \sqrt{x}$

always numbers for the outer integral

Inner: $\int_{y^2}^{2y} 4y+2\,dx$

$= (4y+2)\,x \Big|_{x=y^2}^{x=2y}$

$= (4y+2)(2y-y^2)$

Then integrate

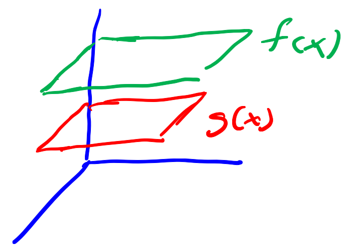
PROPERTIES.

1. If $f(x,y) \geq 0$ for all (x,y) in $\mathcal{R}$, then $\displaystyle\iint_{\mathcal{R}} f(x,y)\; dA$ is the volume of the solid under the graph of $z = f(x,y)$ above the region $\mathcal{R}$.

2. If $f(x,y)$ and $g(x,y)$ are both continuous on the region $\mathcal{R}$, then

$$\iint_{\mathcal{R}} \{f(x,y) \pm g(x,y)\}\; dA = \iint_{\mathcal{R}} f(x,y)\; dA \;\pm \iint_{\mathcal{R}} g(x,y)\; dA.$$

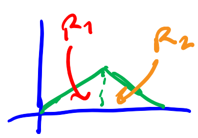
3. For any constant c, $\displaystyle\iint_{\mathcal{R}} cf(x,y)\; dA = c\iint_{\mathcal{R}} f(x,y)\; dA.$



4. If $f(x,y) \geq g(x,y)$, then $\displaystyle\iint_{\mathcal{R}} f(x,y)\; dA \geq \iint_{\mathcal{R}} g(x,y)\; dA.$

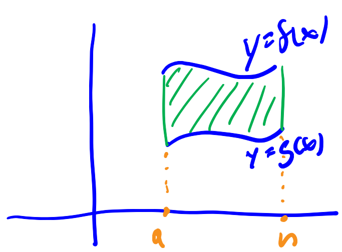
5. If $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, where $\mathcal{R}_1$ and $\mathcal{R}_2$ don't overlap except perhaps on their boundaries, then

$$\iint_{\mathcal{R}} f(x,y)\; dA = \iint_{\mathcal{R}_1} f(x,y)\; dA \;+ \iint_{\mathcal{R}_2} f(x,y)\; dA.$$



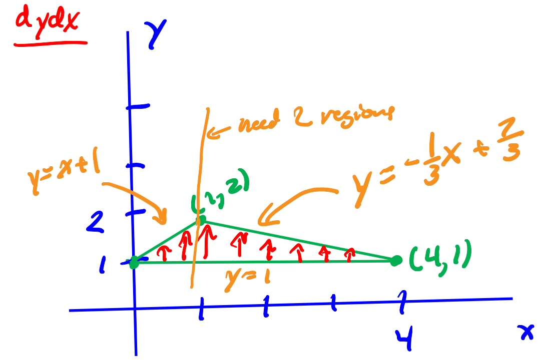
6. If we integrate the constant function $f(x,y) = 1$ over a region $\mathcal{R}$, we get the area of $\mathcal{R}$:

$$\iint_{\mathcal{R}} 1\; dA = A(\mathcal{R}).$$



$$\text{area} = \int_a^b f(x) - g(x)\, dx$$
$$= \int_a^b \left(y \Big|_{y=g(x)}^{y=f(x)} \right) dx$$
$$= \int_a^b \left(\int_{g(x)}^{f(x)} 1\, dy \right) dx = \iint_R 1\, dA$$

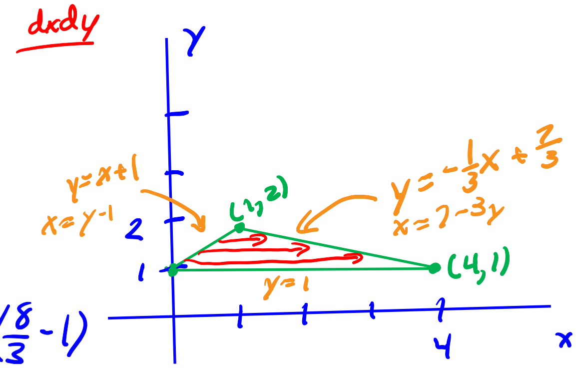
Ex4. Compute $\iint_{\mathcal{R}} f(x,y)\, dA$ if $f(x,y) = y^2$, where $\mathcal{R}$ is the triangular region with vertices $(0,1)$, $(1,2)$ and $(4,1)$.



dydx: ①
$$\iint_R f(x,y)\,dA = \int_0^1\left(\int_{y=1}^{x+1} y^2\,dy\right)dx + \int_1^4\left(\int_{y=1}^{\frac{7-x}{3}} y^2\,dy\right)dx$$

or

dxdy: ②
$$\iint_R f(x,y)\,dA = \int_1^2\left(\int_{x=y-1}^{7-3y} y^2\,dx\right)dy$$



Using (dxdy) ②

Inner: $\displaystyle\int_{y-1}^{7-3y} y^2\,dx = y^2 x\Big|_{x=y-1}^{x=7-3y} = y^2(7-3y) - y^2(y-1)$

$\Rightarrow 7y^2 - 3y^3 - y^3 + y^2 = \boxed{8y^2 - 4y^3}$

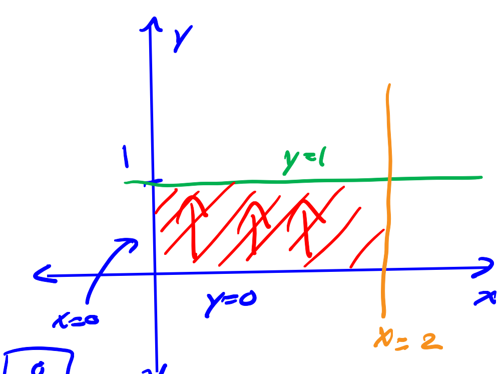
Outer: $\displaystyle\int_1^2 (8y^2 - 4y^3)\,dy = \left(\frac{8}{3}y^3 - y^4\right)\Big|_1^2 = \left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 1\right)$

$= \frac{56}{3} - 15.$

So, $\displaystyle\iint_R f(x,y)\,dA = \frac{11}{3}.$

Ex5. Compute $\iint_{\mathcal{R}} f(x,y)\, dA$ if $f(x,y) = 3 - x^2 - y^2$ and $\mathcal{R}$: $0 \le x \le 2,\ 0 \le y \le 1$.

$$\iint_R f(x,y)\,dA = \int_0^2\left(\int_{y=0}^{y=1} (3 - x^2 - y^2)\,dy\right)dx$$



Inner: $\displaystyle\int_0^1 (3 - x^2 - y^2)\,dy = \left(3y - x^2 y - \frac{y^3}{3}\right)\Big|_{y=0}^{y=1}$

$= \left(3 - x^2 - \frac{1}{3}\right) - (0) = \frac{8}{3} - x^2.$

Outer: $\displaystyle\int_0^2 \left(\frac{8}{3} - x^2\right)dx = \left(\frac{8}{3}x - \frac{x^3}{3}\right)\Big|_0^2 = \left(\frac{16}{3} - \frac{8}{3}\right) - (0-0) = \boxed{\frac{8}{3}}$

So, $\displaystyle\iint_R f(x,y)\,dA = \frac{8}{3}$

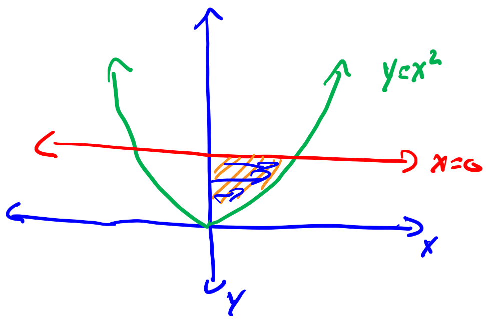
Ex6. Evaluate the following integrals:

(a) $\int_0^1 \int_{x^2}^1 x^3 \sin(y^3) \; dy \; dx = \int_0^4 \left(\int x^3 \sin(y^3)\, dx \right) dy$

$\int x^3 \sin(y^3)\, dy$

$\Rightarrow x^3 \int \sin(y^3)\, dy$

not possible with what we know



$\begin{cases} 0 \le x \le 1 \\ \underbrace{x^2 \le y \le 1}_{y=x^2 \quad y=1} \end{cases}$

Inner: $\int_0^{\sqrt{y}} x^3 \sin(y^3)\, dx = \sin(y^3) \frac{x^4}{4} \Big|_{x=0}^{x=\sqrt{y}} = \frac{y^2 \sin(y^3)}{4} - 0.$

Outer: $\int_0^1 \left(\frac{y^2}{4} \sin(y^3) \right) dy = \frac{-\cos(y^3)}{12} \Big|_{y=0}^{y=1}$

$= \frac{-\cos(1)}{12} - \left(-\frac{\cos(0)}{12}\right)$

so, $\int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\, dx\, dy$

$= \frac{1-\cos(1)}{12}$

$\int \frac{y^2}{4} \sin(y^3)\, dy$ $\quad u = y^3$, $du = 3y^2\, dy$

$\Rightarrow \int \frac{\cancel{y^2}}{4} \frac{\sin(u)}{3\cancel{y^2}}$

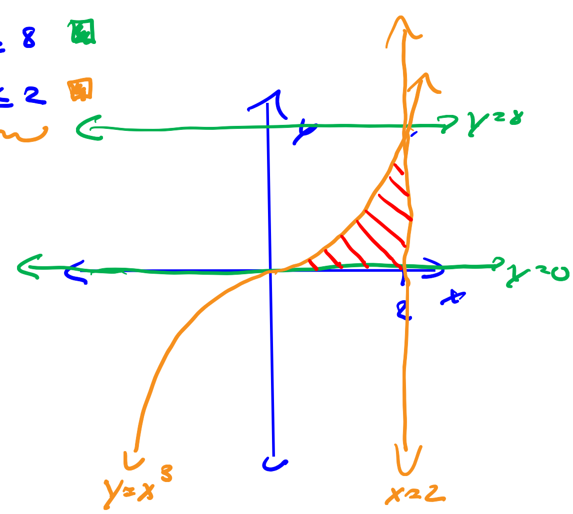
$\Rightarrow \frac{1}{12} \int \sin(u)\, du$

$\Rightarrow \frac{-\cos(u)}{12} + C$

(b) $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4} \; dx \; dy.$

$= \int_0^2 \left(\int_0^{x^3} e^{x^4} dy \right) dx$

$\begin{cases} 0 \le y \le 8 \\ \underbrace{\sqrt[3]{y} \le x \le 2} \end{cases}$



Inner: $\int_0^{x^3} e^{x^4} dy = e^{x^4} y \Big|_{y=0}^{y=x^3} = e^{x^4} \cdot x^3 - 0$

Outer: $\int_0^2 (x^3 e^{x^4})\, dx = \frac{e^{x^4}}{4} \Big|_{x=0}^{x=2} = \frac{e^{16} - e^0}{4}$

so, $\int_0^2 \int_0^{x^3} e^{x^4} dy\, dx = \boxed{\frac{e^{16} - 1}{4}}$

aside

$\int x^3 e^{x^4} dx = \int \cancel{x^3} e^u \frac{du}{4\cancel{x^3}}$

$u = x^4$ $\quad \Rightarrow \int \frac{e^u}{4} du$

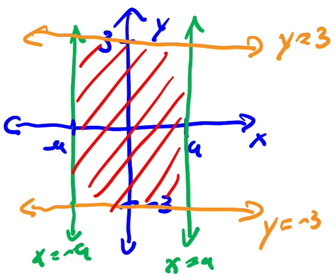
$du = 4x^3 dx$ $\quad = \frac{e^u}{4} + C$

Volume under the graph of a nonnegative function.

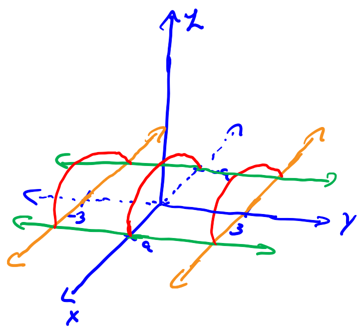
Ex7. Let a be a positive number. Use geometry to evaluate the following double integral

$$\iint_{\mathcal{R}} \sqrt{a^2 - x^2}\, dy\, dx$$

where $\mathcal{R}$ is the rectangle $\underbrace{[-a, a]}_{x} \times \underbrace{[-3, 3]}_{y}$



$$\begin{cases} -a \le x \le a \\ -3 \le y \le 3 \end{cases}$$



$f(x,y) = \sqrt{a^2 - x^2} \ge 0$

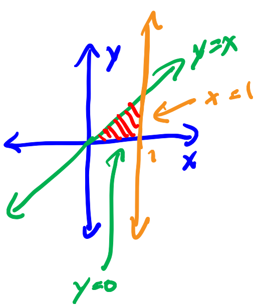
$z = \sqrt{a^2 - x^2}, \; z \ge 0$

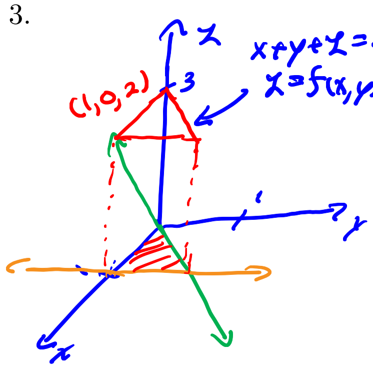
$z^2 = a^2 - x^2, \; z \ge 0$

$x^2 + z^2 = a^2, \; z \ge 0$

$$\iint_R \sqrt{a^2 - x^2}\, dy\, dx = \frac{1}{2}\pi (\text{radius})^2 \cdot (\text{height})$$
$$= \frac{1}{2}\pi (a)^2 \cdot (6)$$
$$\boxed{= 3\pi a^2.}$$

Ex8. Set up an iterated integral for the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies on the plane $x + y + z = 3$.





$x+y+z=3$

$z=f(x,y)=3-x-y$

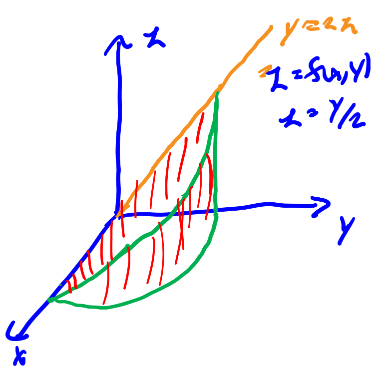
Volume: $\iint_R f(x,y)\, dy\, dx$

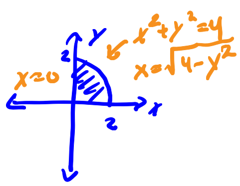
$$\Rightarrow \int_0^1 \int_0^x (3-x-y)\, dy\, dx$$

$= \cdots = 1$

practice: evaluate and check this answer

Ex9. Set up an iterated integral for the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y = 2z$, $x = 0$, $z = 0$ in the first octant.





$$\text{Volume: } \int_0^2 \int_0^{\sqrt{4-y^2}} \underbrace{\left(\frac{y}{2}\right)}_{f(x,y)} dx\, dy$$

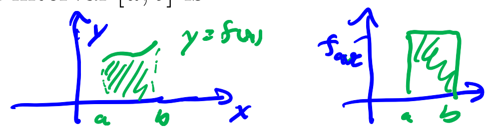
$= \cdots = \frac{4}{3}$

check for practice

Average Value of a Function.

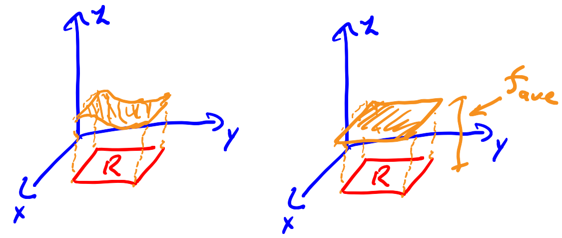
Recall that the average value of a function f of one variable defined on an interval $[a, b]$ is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\ dx$$



In a similar fashion we define the average value of a function f in two variables defined on a region $\mathcal{R}$ to be

$$f_{ave} = \frac{1}{A(\mathcal{R})}\iint_{\mathcal{R}} f(x,y)dA$$



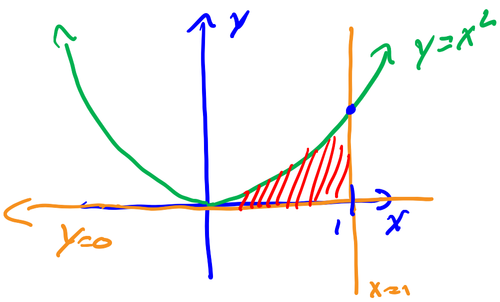
where $A(\mathcal{R})$ is the area of the region $\mathcal{R}$.

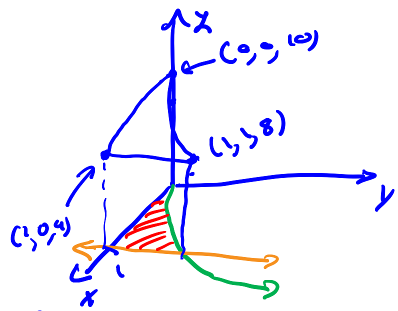
If $f(x, y) \geq 0$, the equation

$$A(\mathcal{R}) \cdot f_{ave} = \iint_{\mathcal{R}} f(x,y)dA$$

says that the cylinder with base $\mathcal{R}$ and height f_{ave} has the same volume as the solid that lies under the graph of the function $z = f(x, y)$.

Ex10. Find the average value of $f(x, y) = 10 - x^2 - y^2$ over the region $\mathcal{R}$ enclosed by the curves $y = 0$, $y = x^2$ and $x = 1$.





•) area of $R = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$.

•) $\iint 10 - x^2 - y^2\ dA = \int_0^1 \int_0^{x^2} (10 - x^2 - y^2)\,dy\,dx$

Inner: $\int_0^{x^2} 10 - x^2 - y^2\,dy = \left(10y - x^2 y - \frac{y^3}{3}\right)\Big|_{y=0}^{y=x^2}$

$= 10x^2 - x^4 - \frac{x^6}{3} - (0 - 0 - 0)$

Outer: $\int_0^1 10x^2 - x^4 - \frac{x^6}{3}\,dx = \cdots = \frac{10}{3} - \frac{1}{5} - \frac{1}{21}$.

so, $f_{ave} = \dfrac{\frac{10}{3} - \frac{1}{5} - \frac{1}{21}}{1/3} = \dfrac{324}{35}$

TO-DO:

(1) Let $g(x,y)$ be a continuous function. Sketch the region of integration and then reverse the order of integration for

$$\int_0^1 \int_{\sqrt{y}}^{2-y} g(x,y)\ dx\ dy.$$

(2) Consider $\displaystyle\iint_{\mathcal{R}} x^2+y^2\,dA$, where $\mathcal{R}$ is the region bounded by $y=2x$ and $y=x^2$.
Sketch the region of integration $\mathcal{R}$ and set up iterated integrals for both orders of integration.

(3) The value of $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{4+y^3}\ dy\ dx$ equals

(a) $\ln(3)$

(b) $\dfrac{\ln(3)}{3}$

(c) $\ln(12)$

(d) $\dfrac{\ln(12)}{3}$

(4) Set up an iterated integral to compute $\displaystyle\iint_{\mathcal{R}} (x+2y)\ dA$ where $\mathcal{R}$ is the region bounded by $y=x^2$ and $y=4-x^2$.